TWO THEOREMS IN THE GEOMETRY OF CONTINUOUSLY TURNING CURVES.

By PROFESSOR O. D. KELLOGG, University of Missouri.

The simple theorems given in the following pages were found necessary in investigating the continuity of certain line integrals occurring in the theory of logarithmic potentials.* They are thought to be interesting examples of the arithmetization of geometric theorems which are by no means self evident but which admit of an easy analytic proof.

We shall be concerned with a single closed curve, C, without double points and with continuously turning tangent.

Arithmetically. We shall be concerned with a pair of functions, $x(s)$, $y(s)$, such that,

(a) $x(s+nl)=x(s)$, $y(s+nl)=y(s)$, $x'(s+nl)=x'(s)$, $y'(s+nl)=y'(s)$, where l is constant and n is any integer;

(b) $x'(s)$ and $y'(s)$ are continuous;

(c) $x'^2(s)+y'^2(s)=1$;

(d) The equations $\left.\begin{matrix} x(t)=x(s) \\ y(t)=y(s) \end{matrix}\right\}$ have no simultaneous solution, (t, s), except $t=s+nl$.

From the above hypotheses follows the uniform continuity of $x(s)$, $y(s)$, $x'(s)$, $y'(s)$.

We pass now to the theorems, giving them first a geometric form, then changing this to the arithmetic form and supplying the proof.

Theorem I. *It is possible to describe about any point of the curve, C, a circle which shall cut from C at most one segment, namely, that passing through the center of the circle; moreover the same radius may be made to serve for the circle at all points of C.*

Let us write

$$\rho\,(t, s)=\sqrt{[x(t)-x(s)]^2+[y(t)-y(s)]^2},$$

where here and always in the following, the positive square root is meant.

The function $\rho\,(t, s)$ is continuous in both variables; it therefore attains its maximum, R, say, for $t=t_m$, $s=s_m$. Choose $0<r<R/2$. Then for fixed t, the equation $\rho\,(t, s)=r$ has a solution; for $\rho\,(t, t)=0$, and as

$$\rho\,(t, s_m)+\rho\,(t, t_m)>(t_m, s_m)=R,$$

either $\rho\,(t, s_m)$ or $\rho\,(t, t_m)$ is greater than $R/2$ and hence than r. Thus the

*See "Potential Functions on the Boundary of their Regions of Definition," *Transactions of the American Mathematical Society*, Vol. 9, No. 1, pp. 43 and 49.

continuous function $\rho\ (t, s)-r$, which is positive for $s=t$ and negative at either $s=t_m$ or $s=s_m$ must vanish at some intermediate point.

Calling $s-t=h$, the equation $\rho\ (t, t+h)=r$ has a first positive solution $h=P(t, r)$, and a first negative solution $h=N(t, r)$. Our theorem I may now be stated arithmetically:

It is possible to choose r, independent of t, so small that the inequality,

$$\rho\ (t, s)<r, \tag{1}$$

holds only for values of s and t satisfying the inequalities

$$\left.\begin{array}{r} N(t, r)<s-t+nl<P(t, r) \\ i.\ e.,\ N(t, r)<h+nl<P(t, r) \end{array}\right\} \tag{2}$$

Passing to the proof, we observe first of all that $\rho\ (t, N(t, r))=r$ and $(t, \rho(t, P(t, r)))=r$, and we shall be concerned with choosing r, independently of t so that, first, as $h+nl$ passes out from the interval $N(t, r)<h+nl<P(t, r)$, $\rho\ (t, t+h)$ becomes greater than r, and secondly, that it remains so.

As for the first, $\rho\ (t, t+h)$ for fixed t, increases with $|h|$. For

$$\frac{\frac{\partial}{\partial h}\rho^2(t, t+h)}{h}=\frac{x(t+h)-x(t)}{h}.x'(t+h)+\frac{y(t+h)-y(t)}{h}.y'(t+h)$$

$$=x'(t+I_1.h)x'(t+h)+y'(t+I_2.h)y'(t+h) \quad (0\leqq I_1\leqq 1,\ 0\leqq I_2\leqq 1).$$

This function approaches 1 as h approaches 0, and as $x'(s)$ and $y'(s)$ are uniformly continuous, we can find a positive quantity, p, such that the function is positive so soon as $|h|<p$, and this independently of t. For h so restricted we have

$$\frac{2\rho}{h}\frac{\partial \rho}{\partial h}>0, \text{ whence } \frac{\partial \rho}{\partial h} \text{ has the sign of } h,$$

so that $\rho(t, t+h)$ increases with $|h|$ for $|h|<p$, as stated. Let now r_1 be the minimum of the continuous function of t, $\rho\ (t, t+p)$, in the closed interval $0\leqq t\leqq 1$. This minimum is actually attained, and cannot therefore be 0 because of hypothesis (d). If now as a first restriction upon r we make it less than r_1, the inequality (1) cannot hold in either of the intervals

$$-p<h<N(t, r), \quad P(t, r)<h<p,$$

and hence in any of the intervals,

$$-p<h+nl<N(t,\ r),\quad P(t,\ r)<h+nl<p.$$

As for the second part of the proof, that the inequality (1) cannot hold outside the interval from $-p$ to p (except for values of h congruent to values within the interval, modulo 1), we shall find it convenient to restrict ourselves to a fundamental region of the periodic function $\rho\ (t,\ s)$, namely, $0 \leqq t \leqq 1$, $0 \leqq s \leqq 1$. Every point $(t,\ s)$ not already considered, that is, not satisfying the inequalities

$$-p<s-t+nl<p$$

has a representative point (for which the function has the same value) in the subregion

$$p \leqq |\ s-t\ | \leqq 1-p,\quad 0 \leqq t \leqq 1,\quad 0 \leqq s \leqq 1.$$

But this is a closed region, and in it $\rho\ (t,\ s)$ cannot vanish, by hypothesis (*d*), the function therefore has a positive minimum r_2. If now r be also taken less than r_2, $\rho\ (t,\ s)>r_2>r$ for all points of the region considered.

It has thus been shown that if $r<r_1$, and $r<r_2$, condition (1) can only hold when t and s satisfy the inequalities (2), as was to be proved.

Theorem II. *If normals be drawn to the curve C at all points of an arc of length 2* α*, and if segments of length* β *be measured off upon these normals to either side of C, a circle can be drawn with center at the mid-point of the arc of C considered, such that the entire surface of the circle will be covered by the segments of the normals; moreover one choice can be made for the radius of the circle which will hold for all situations of the arc upon C.*

Arithmetically this may be stated: *Given* α *and* β*, we can find r, independent of t, and such that for any point* $(\xi,\ \eta)$ *satisfying the condition*

$$\rho(\xi,\ \eta;\ t)=\sqrt{\{[\xi-x(t)]^2+[\eta-y(t)]^2\}}<r,$$

the equations

$$\xi=x(t+h)-\lambda\, y'(t+h), \tag{3}$$
$$\eta=y(t+h)-\lambda\, x'(t+h), \tag{4}$$

admit of at least one solution, $(h,\ \lambda)$*, where* $|\ h\ |<\alpha$ and $|\ \lambda\ |<\beta$.

Evidently if proved for one α and β, the theorem holds for every greater pair of values. We shall therefore replace them by a single number, $2n$, smaller than either, and also less than p of the preceding paragraphs. Then r satisfies the requirements of the theorem if

$$r<r_3/3, \tag{5}$$

where r_3 is the minimum of $\rho(t,\ t+n)$. To prove it we consider the function

$$\rho^2(\xi,\ \eta;\ t+h)-r_3^2/4.$$

This is a continuous function of h, and for $h=0$ it is negative because of (3) and (5). On the other hand, it is positive for $h=n$ and also for $h=-n$, as we proceed to show. By the definition of r_3, $\rho(t,\ t+n)\geqq r_3$, and since

$$\rho(\xi,\ \eta;\ t)+\rho(\xi,\ \eta;\ t+n)>\rho(t,\ t+n)$$

we have

$$\rho(\xi,\ \eta;\ t+n)>r_3-\rho(\xi,\ \eta;\ t),$$

and as by conditions (3) and (5), $\rho(\xi,\ \eta;\ t)<r<r_3/3$, we have

$$\rho(\xi,\ \eta;\ t+n)>2r_3/3.$$

Hence

$$\rho^2(\xi,\ \eta;\ t+h)-r_3^2/4>0,$$

for $h=n$, and similarly for $h=-n$.

It follows that this function vanishes at some point $h=h'$ between 0 and n, and also at some point $h=h''$ between 0 and $-n$. But its derivative is also continuous and hence, by Rolle's theorem, vanishes at some point between $h=h'$ and $h=h''$, say $h=\bar{h}$:

$$[\xi-x(t+\bar{h})]x'(t+\bar{h})+[\eta-y(t+\bar{h})]y'(t+\bar{h})=0.$$

As $x'^2(t+\bar{h})+y'^2(t+\bar{h})=1$, both $x'(t+\bar{h})$ and $y'(t+\bar{h})$ do not vanish; to fix ideas, suppose the first is different from 0. Putting the ratio

$$\frac{\eta-y(t+\bar{h})}{x'(t+\bar{h})}\ \text{equal to}\ \bar{\lambda},$$

we have

$$\eta=y(t+\bar{h})+\bar{\lambda}x'(t+\bar{h}),\quad \xi=x(t+\bar{h})-y'(t+\bar{h}),$$

showing that $\bar{h}$ and $\bar{\lambda}$ are a solution of the equations (4). It is at once clear that $\bar{h}$ obeys the restriction laid upon it. To see the same for $\bar{\lambda}$, transpose in the above equations the first terms on the right, then square and add. The result is

$$\bar{\lambda}=\pm\rho(\xi,\ \eta;\ t+\bar{h}).$$

But

$$\rho(\xi,\ \eta;\ t+\bar{h})\leqq\pi(\xi,\ \eta;\ t)+\rho(t,\ t+\bar{h}),$$

in which $\rho(\xi,\ \eta;\ t)<r<r_3/3$, and because $\rho(t,\ t+h)$ increases as $|h|$ increases, $\rho(t,\ t+\bar{h})<\rho(t,\ t+n)<r_3$,

$$|\bar{\lambda}| < 4r_3/3,$$

that is, since r_3 is the chord belonging to an arc of length n, $|\bar{\lambda}| < 4n/3$, and hence $<2n$, and the theorem is proved.

It does not of course follow from the above that through a given point of the surface of the circle *only one* normal passes. Indeed if part of the curve C be of the form $y=x^{\frac{4}{3}}$, which has infinite curvature at the origin, and yet has a continuously turning tangent, it is clear that there are points as near as we please to the curve through which there are three normals, all belonging to as short an arc of C as we please. A corollary to theorem I, however, supplies us with a uniqueness theorem which is often just as useful in the applications:*

The chords of the circle in theorem I which are perpendicular to the tangent at C at the center of the circle, cut C but once. In other words, if instead of taking a little field of normals about a point of C, we take a field of straight lines through the points of a limited arc of C and parallel to the normal to C at the mid-point of the arc, these lines will cover just once the surface of a little circle.

The equations of one of these parallels to the normal to C through the point $x(t)$, $y(t)$ will be

$$\begin{aligned}\xi&=x(t)+\delta x'(t)-\mu y'(t),\\ \eta&=y(t)+\delta y'(t)+\mu x'(t),\end{aligned}$$

where δ is the distance of the parallel from $x(t)$, $y(t)$, and μ is a parameter. Arithmetically therefore, the theorem may be stated:

The equations

$$\begin{aligned}x(t+h)&=x(t)+\delta x'(t)-\mu y'(t),\\ y(t+h)&=y(t)+\delta y'(t)+\mu x'(t),\end{aligned}$$

admit of only one solution for which $\rho(t, t+h)<r$.

To see this, eliminate μ. We find that we have merely to show that the function

$$F(h)=x'(t)[x(t+h)-x(t)]+y'(t)[y(t+h)-y(t)]-\delta,$$

has at most one solution for which $\rho(t, t+h)<r$. But

$$F'(h)=x'(t)x'(t+h)+y'(t)y'(t+h),$$

which we have previously seen to be positive for $|h|<p$, and this includes

*See Liapounoff, Sur certaines questions qui se rattachent au probleme de Dirichlet; Journal de Math. Liouville, 1898. The author makes an hypothesis upon his bounding surface which the above reasoning, carried out for space, would show to be unnecessary.

all values of h corresponding to $\rho(t, t+h)<r$, and hence $F(h)$, being an always increasing function in the interval, can vanish at most but once. We omit the simple proof that if $\delta<r$ and if r is small enough $F(h)$ *does* vanish once.

DIVISIONS OF AN ANGLE INTO EQUAL PARTS BY MEANS OF A TRANSCENDENTAL CURVE.

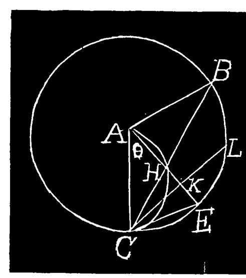
By J. S. BROWN, San Marcos, Texas.

The problem of trisecting an angle by means of the cardioid is capable of a much more general setting as follows:

Given CAE an isosceles triangle, $AC=AE$, angle $A=\theta$; also given a point H in AE such that angle $HCE=n.\theta$. Required, the locus of H if AC is fixed and AE rotates in the plane about A. E will describe a circle with A as center and AE as radius, as indicated in the figure.

Draw CL perpendicular to AE; angle $LCE=\frac{1}{2}\theta$; $CK=CH\cos(n-\frac{1}{2})\theta$, and $CK=AC\sin\theta$. Then

$$CH=\frac{AC\sin\theta}{\cos(n-\frac{1}{2})\theta}, \tag{1}$$



which is the polar equation of the locus of H when C is taken as origin and AC is the axis of reference.

By assigning the successive values $\frac{1}{2}$, 1, $1\frac{1}{2}$, etc., to n the particular equations of the curves by means of which subdivisions one-half, one-third, etc., of an angle can be constructed, are found.

As an example, suppose it is desired to find the equation of the curve by means of which an angle with A as vertex and AC as one side, can be trisected; for example, angle BAC.

It is evident that in this case $n=1$. Substituting this value of n in the general equation, we find

$$CH=\frac{AC\sin\theta}{\cos\frac{1}{2}\theta}=\frac{AC.2\sin\frac{1}{2}\theta.\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}=AC.2\sin\frac{1}{2}\theta=2AC\sin\frac{1}{2}\theta. \tag{2}$$

Suppose AHC to be the curve for this value of n, and let H be the point where BC cuts this curve. Draw AHE. The angle $CAE=\frac{1}{3}CAB$, for angle $BCE=\theta$ and angle $BAE=2\theta$.

Evidently, this curve corresponding to (2) is the cardioid, though the